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LETTER TO THE EDITOR

The massless scalar field around a static black hole

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Abstract. An explicit expression for a massless scalar field of a point charge resting near a static black hole is obtained, and the properties of this solution are described.

There is now a strong belief that the static black hole which appears as a result of gravitational collapse can be uniquely specified by its mass and electric charge, while the other characteristics fade away during the collapse process. In particular, the static black hole cannot possess the massless scalar field 'hairs' (Chase 1970). If a body with a small scalar charge is dropped into a static black hole and approaches the event horizon, then in principle two different outcomes seem plausible: (i) the scalar field increases and its stress energy may destroy the event horizon; (ii) the event horizon will not be destroyed. To investigate this problem and describe the detailed structure of the scalar field, one needs to solve the scalar field equation with a point-like source in a given black-hole geometry. As far as we know, the corresponding exact solution has not been found. (See however the papers by Chase (1970) and Rowan and Stephenson (1976), where the series representation for a solution is given.)

In the case of the electromagnetic field the exact solution has been found and the corresponding field structure investigated in great detail by Copson (1928), Cohen and Wald (1971), Hanni and Ruffini (1973), Linet (1976) and Leaute and Linet (1976). In this paper we find an explicit solution for a massless scalar field of a point charge which is slowly lowered into Schwarzschild and Reissner–Nordström black holes.

We consider the massless scalar field Φ obeying the equation

$$\Box\Phi = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial\Phi}{\partial x^{\nu}}\right) = \rho \tag{1}$$

in a background of the metric

$$ds^2 = -f(r)\,dt^2 + f^{-1}(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2). \tag{2}$$

In the particular case of the Reissner–Nordström black hole $f(r) = 1 - 2m/r + e^2/r^2$, $(G = c = 1)$. Here ρ is the density of the scalar field static source. For a point-like scalar charge g resting at a point $r = b$, $\theta = \theta_b$, $\varphi = \varphi_b$,

$$\begin{aligned}\rho(x) &= g\int d\tau\,\delta^4(x, x(\tau)) \\ &= g\frac{\delta(r-b)\delta(\theta-\theta_b)\delta(\varphi-\varphi_b)}{r^2|\sin\theta|}f^{1/2}(r).\end{aligned} \tag{3}$$

The equation (1) in the metric (2) can be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}\right]=\rho. \tag{4}$$

If we use for a field Φ the representation

$$\Phi=\hat{F}[W],$$
$$\hat{F}[W]=-\int_r^\infty\frac{\mathrm{d}r\,W(r,\theta,\varphi)}{rf(r)}, \tag{5}$$

then the function $W(r,\theta,\varphi)$ satisfies the equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial W}{\partial r}\right)+\frac{1}{r^2 f}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial W}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 W}{\partial\varphi^2}\right]=\frac{1}{r}\frac{\partial}{\partial r}(r^2\rho). \tag{6}$$

It can be easily verified that this equation is exactly the same as the equation for an electric potential of a static charge distribution $j^\mu=(j^0,0,0,0)$, $j^0=-\partial_r(r^2\rho)/r$. In particular, the above constructed linear operator $\hat{F}$ transforms the static homogeneous solution for the electromagnetic potential into a static homogeneous solution for a massless scalar field equation. The operator $\hat{F}$ evidently possesses an inverse.

This remarkable property of the scalar field equation allows us to obtain the expression for a field $\Phi_b(r,\theta,\varphi)$ created by a scalar charge placed at a point $r=b$, $\theta=\theta_b$, $\varphi=\varphi_b$ in an explicit form, if the corresponding potential $V_b(r,\theta,\varphi)$ of an electric point charge

$$j^0=g\frac{\delta(r-b)\delta(\theta-\theta_b)\delta(\varphi-\varphi_b)}{r^2|\sin\theta|}$$

is known:

$$\Phi_b(r,\theta,\varphi)=-f^{1/2}(b)\frac{\partial}{\partial b}\left[b\int_r^\infty\frac{\mathrm{d}r}{rf(r)}V_b(r,\theta,\varphi)\right]. \tag{7}$$

Using the expression for $V_b(r,\theta,\varphi)$ in a Reissner–Nordström metric obtained by Leaute and Linet (1976),

$$V_b(r,\theta,\varphi)=\frac{g}{4\pi rb}$$
$$\times\left(\frac{(r-m)(b-m)-(m^2-e^2)\lambda}{[(r-m)^2+(b-m)^2-2(r-m)(b-m)\lambda-(m^2-e^2)(1-\lambda^2)]^{1/2}}+m\right),$$
$$\lambda=\cos\theta\cos\theta_b+\sin\theta\sin\theta_b\cos(\varphi-\varphi_b), \tag{8}$$

we obtain

$$\Phi_b(r,\theta,\varphi)=\frac{g}{8\pi}\left(1-\frac{2m}{b}+\frac{e^2}{b^2}\right)^{1/2}(A(\infty)-A(r)), \tag{9}$$

where

$$A(r)=\left(1+\frac{\beta-\rho\lambda}{R}\right)\left[\left(\beta-\mu\lambda-\frac{(\beta\lambda-\mu)(\rho-\mu)}{\beta-\mu\lambda+R}\right)^{-1}+\left(\beta+\mu\lambda-\frac{(\beta\lambda+\mu)(\rho+\mu)}{\beta+\mu\lambda+R}\right)^{-1}\right],$$
$$A(\infty)\equiv A(r=\infty)=\frac{1}{\beta+\mu}+\frac{1}{\beta-\mu}$$

and

$$\beta = b - m, \qquad \rho = r - m, \qquad \mu = (m^2 - e^2)^{1/2},$$
$$R^2 = \rho^2 + \beta^2 - 2\rho\beta\lambda - \mu^2(1 - \lambda^2).$$

It should be noted that the scalar curvature for the Reissner–Nordström geometry vanishes. Thus in the case under consideration the conformal scalar field equation coincides with the equation (1).

The obtained solution (9) possesses the following properties. If $b > 2m$ then it is regular everywhere except the evident pole at the point source. For $m + (m^2 - e^2)^{1/2} < b < 2m$ an additional pole lying inside the black hole at a point $r = 2m - b$, $\theta = \pi - \theta_b$, $\varphi = \varphi_b + \pi$ arises. The residue of Φ_b at this pole is $(g/4\pi)(2m - b)/b$. The field Φ_b and the invariant $|\nabla\Phi_b|^2$ are finite at the event horizon. The asymptotic behaviour of Φ_b at large distances ($r \to \infty$, b is fixed) is

$$\Phi_b(r, \theta, \varphi) = -\frac{g}{4\pi}\frac{1}{r}\left(1 - \frac{2m}{b} + \frac{e^2}{b^2}\right)^{1/2}. \qquad (10)$$

If the point charge tends to the event horizon $b \to r_+$, $r_+ = m + (m^2 - e^2)^{1/2}$ then the field Φ_b at a fixed point $\boldsymbol{r}$ tends to zero as $(b - r_+)^{1/2}$. The obtained solution (9) allows one to write the solution of equation (1) with an arbitrary charge density distribution $\rho(\boldsymbol{r})$ in the form

$$\Phi(r, \theta, \varphi) = \int \mathrm{d}\boldsymbol{b}\; \Phi_b(r, \theta, \varphi)\rho(\boldsymbol{b}). \qquad (11)$$

The representation (11) shows that the conclusion about the fading away property of the scalar field also remains valid when a distributed scalar charge is falling into a black hole.

These results allow us to conclude that when a test scalar charge is slowly lowered into a static black hole, the scalar field fades away and the black hole will not be destroyed.

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